Girraween High School

Year 12 Mathematics Extension 2
Task 2, April-2025

General Instructions

- Working Time - 90 minutes + 5 minutes reading time
- Write using black pen only
- Calculators approved by NESA may be used
- For questions in Section II, Show relevant mathematics reasoning and/ or calculations

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Total Marks: 68

Section I - 5 marks (page 1 & 2)

- Attempt questions 1-5
- Allow about 10 minutes for this section

Section II - 63 marks (pages 3-8)

- Attempt questions 6-11
- Allow about 80 minutes for this section

Question 1 (1 mark)

Which of the following is a symbolic representation of the statement:

"At least one of your friends is perfect"

Let $P(x)$ be "x is perfect" and let "F(x)" be "x is your friend" and let the domain be all people.

A. $\forall\, x(F(x) \rightarrow P(x))$

B. $\forall\, x(F(x) \cap P(x))$

C. $\exists\, x(F(x) \cap P(x))$

D. $\exists\, x(F(x) \rightarrow P(x))$

Question 2 (1 mark)

Consider the statement:

"For any function f(x), f(x) has a stationary point of inflexion at $x = c \implies f''(c) = 0$"

A. The contapositive statement is false and the converse is false.

B. The contapositive statement is false and the converse is true.

C. The contapositive statement is true and the converse is false.

D. The contapositive statement is true and the converse is true.

Question 3 (1 mark)

The angles made between $\underset{\sim}{v} = \overrightarrow{OV}$ where v is the point $(5, -3, 3)$ with the positive direction of x, y and z axea respectively are:

A. $40°19'$, $117°14'$ and $62°46'$

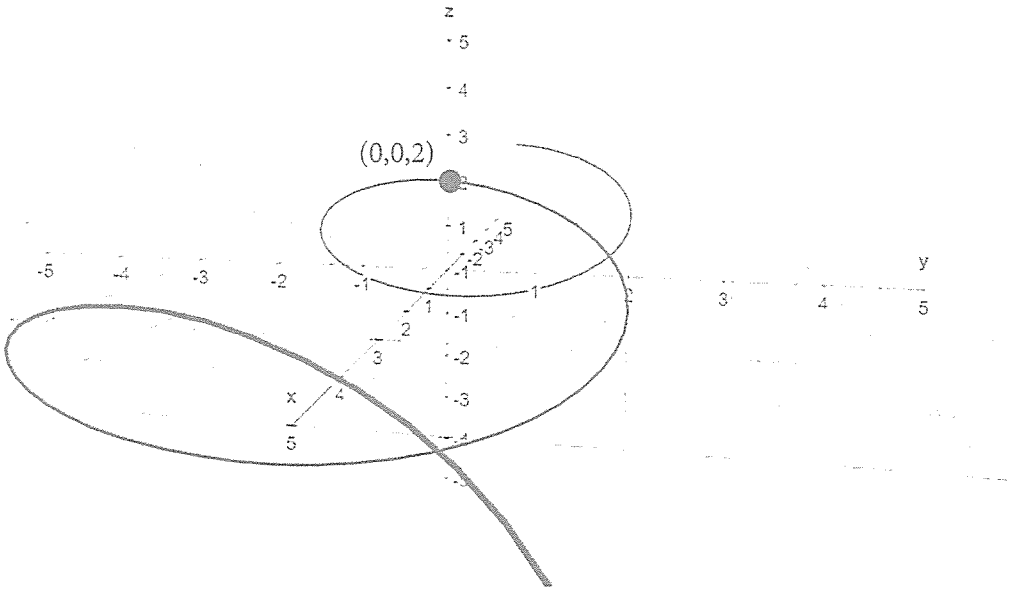
B. $40°19'$, $62°14'$ and $62°46'$

C. $180°$, $53°8'$ and $53°8'$

D. $0°$, $126°52'$ and $53°8'$

Question 4 (1 mark)

What is the possible equation for the graph below?



A. $u = (2\sin t, t, 2\cos t)$

B. $u = (t, 2\sin t, 2\cos t)$

C. $u = (2\cos t, t, 2\sin t)$

D. $u = (t, 2\cos t, 2\sin t)$

Question 5 (1 mark)

The points R, S and T are collinear.

Given that $\overrightarrow{OR} = \underset{\sim}{i} + \underset{\sim}{j}$, $\overrightarrow{OS} = 2\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}$ and $\overrightarrow{OT} = 3\underset{\sim}{i} + m\underset{\sim}{j} + n\underset{\sim}{j}$, the values of m and n are:

A. $m = 3$, $n = -2$

B. $m = -1$, $n = 0$

C. $m = 0$, $n = 1$

D. $m = -3$, $n = 2$

Mathematics Extension 2

Section II

63 Marks

Attempt Questions 6-11

Allow about 80 minutes for this section

Question 6 (9 marks)

(a) Consider the statement: [1]

$$\forall x \in \mathbb{R}, (x > 1) \implies (x < x^2)$$

Write the negation.

(b) Use proof by contradiction to prove that if a, b are integers, then $a^2 - 4b - 3 \neq 0$. [3]
You may wish to consider the case when a is even and the case when a is odd.

(c) Prove that $\log_2 n$ is irrational when n is an odd integer greater than or equal to 3. [3]

(d) Provide a non-inductive proof as to why $12^n > 5^n + 7^n$ for all integers $n \geq 2$. [2]

Question 7 (8 marks)

(a) Use the method of proof by contradiction to prove that the cube root of any prime numbers p, where $p \geq 2$, is irrational. [3]

(b) If a, b, c are real and unequal and that $a^2 + b^2 > 2ab$ decuce that

i. $a^2 + b^2 + c^2 > ab + bc + ac$. [1]

ii. If $a + b + c = 6$ show that $ab + bc + ac < 12$. [2]

(c) Prove that the sum of the squares of three consecutive integers that are even and positive, always gives a remainder 8 when divided by 12. [2]

Question 8 (8 marks)

(a) Prove the following logical equivalence for n a natural number: [4]

$n+6$ is a positive odd integer if and only if $n+21$ is a positive even integer.

Explain your reasoning carefully.

(b) For all non-negative numbers x, and y, $\dfrac{x+y}{2} \geq \sqrt{xy}$ (DO NOT prove this) [2]

A Rectangle has dimensions a and b

Given that the rectangle has perimeter P, and area A, prove that $P^2 \geq 16A$

(c) The prime numbers are 2, 3, 5, 7, 11, 13, 17... [2]

The square numbers are 1, 4, 9, 16, ...

With one exception,

"if a square number differs from a prime by 1, the prime is bigger."

For example, 16 is a square and 17 is a prime. The prime is bigger. The only exception is prime 3 and square 4.

It is conjectured that for all consecutive numbers that are a prime and a square, except for 3 and 4, the prime is bigger.

Prove the conjecture is true.

Question 9 (13 marks)

(a) Given the vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$ satisfy $\underset{\sim}{u}+\underset{\sim}{v} = 17\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$ and $\underset{\sim}{u}-\underset{\sim}{v} = \underset{\sim}{i} + 9\underset{\sim}{j} - 4\underset{\sim}{k}$, find the acute angle between the vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$. [3]

(b) D is a midpoint of the side BC of a triangle ABC. Using vectors, show that: [4]

$$|\overrightarrow{AB}|^2 + |\overrightarrow{AC}|^2 = 2(|\overrightarrow{AD}|^2 + |\overrightarrow{BD}|^2)$$

(c) With respect to a fixed origin O, the lines L_1 and L_2 have equations

$$r_1 = \begin{pmatrix} 11 \\ 2 \\ 17 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 1 \\ -4 \end{pmatrix}$$

and

$$r_2 = \begin{pmatrix} -5 \\ 11 \\ p \end{pmatrix} + \mu \begin{pmatrix} q \\ 2 \\ 2 \end{pmatrix}$$

where λ and μ are scalar parameters and p and q are constants.

i. If L_1 and L_2 intersect at right angles, show that $q = -3$ [1]

ii. Find the value of p. [3]

iii. Find the coordinates of the point of intersction. [2]

Question 10 (13 marks)

(a) The point $P(3,2,1)$ lies on the sphere S_1 with centre $C(2,3,4)$. The point Q is located such that PQ is a diameter.

i. Find $Q(a,b,c)$ [1]

ii. Show that the line with vector equation [2]

$$r = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}$$

is a tangent to the sphere S_1

iii. Find the vector equation of the line l through Q which is perpendicular to both PQ and the direction of $\underset{\sim}{r}$ [3]

(b) Consider the parallelogram $ABCD$ shown. The point F lies on BC such that $\overrightarrow{BF} = \frac{1}{3}\overrightarrow{BC}$ and the point E lies on DCsuch that $\overrightarrow{DE} = \frac{3}{4}\overrightarrow{DC}$. The point X is the intersection of $\overrightarrow{DF}$ and $\overrightarrow{AE}$. Let $\overrightarrow{DX} = \lambda\overrightarrow{DF}$ and $\overrightarrow{AX} = \mu\overrightarrow{AE}$ where λ and μ are parameters. Let $\overrightarrow{AB} = \underset{\sim}{p}$ and $\overrightarrow{AD} = \underset{\sim}{q}$.

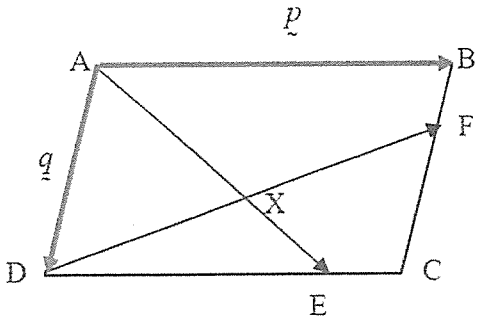


Diagram not to scale.

Copy the diagram.

i. Prove that $\overrightarrow{AX} = \mu\left(\underset{\sim}{q} + \frac{3}{4}\underset{\sim}{p}\right)$ [2]

ii. Hence, or otherwise, prove that $\overrightarrow{AE}$ bisects $\overrightarrow{DF}$ [3]

(c) The curve C is defined in parametric form by:
$x = e^t \cos\, t$ and $y = e^t \sin\, t$, for $t \in R$.

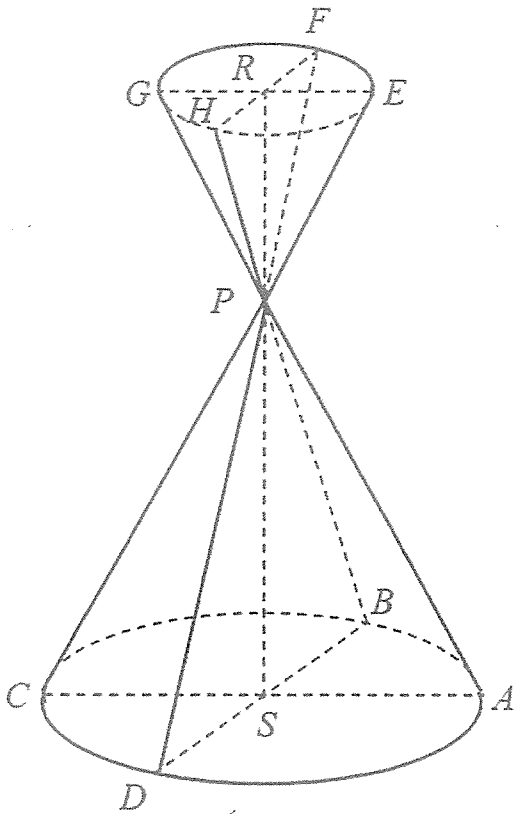
i. Determine the nature of the curve as $t \to \infty$ [1]

ii. sketch the curve. [1]

Question 11 (12 marks)

(a) The diagram shows a small cone on the top of a larger cone. The two cones have a common axis of symmetry passing by the centres R and S of their circular bases and their common apex P. [3]

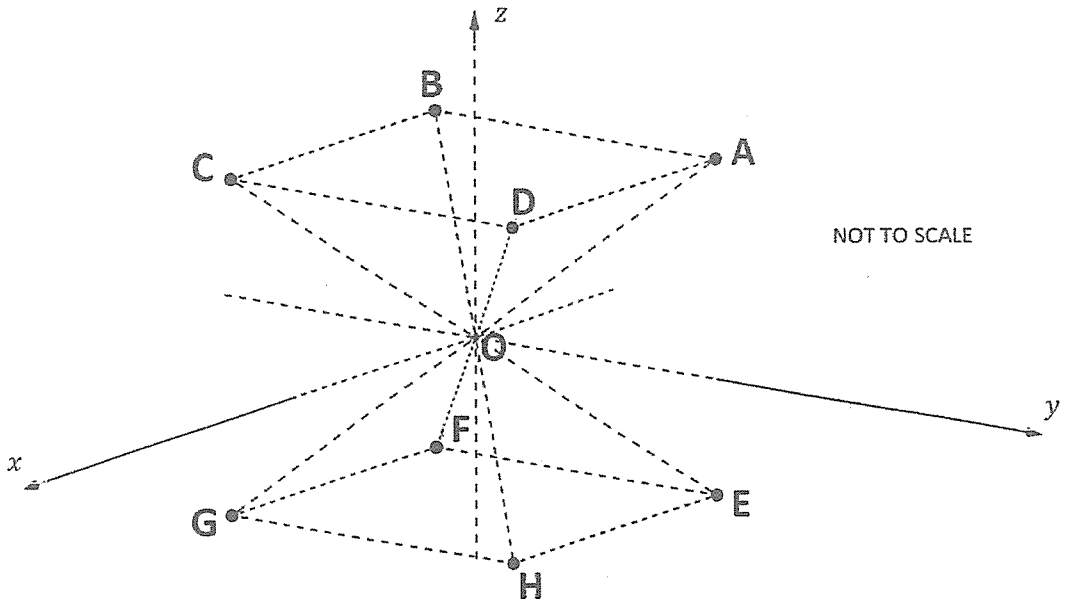
The top cone has a radius r and height h and the bottom cone has radius $2r$ and height $2h$.



Show that $\overrightarrow{PA} + \overrightarrow{PB} + \overrightarrow{PC} + \overrightarrow{PD} + \overrightarrow{PE} + \overrightarrow{PF} + \overrightarrow{PG} + \overrightarrow{PH} = 2\overrightarrow{PS}$

(b) Two square pyramids $OABCD$ and $OEFGH$ are inscribed inside a sphere with centre $O(0, 0, 0)$ and radius 1 unit.

The points A, B, C, D, E, F, G and H all lie on the surface of the sphere and $ABCDEFGH$ forms a cube.



i. Show that the longest diagonal in a cube with side length a units is $a\sqrt{3}$ units. [1]

ii. Hence, derive that $a = \dfrac{2}{\sqrt{3}}$ for the cube $ABCDEFGH$. [1]

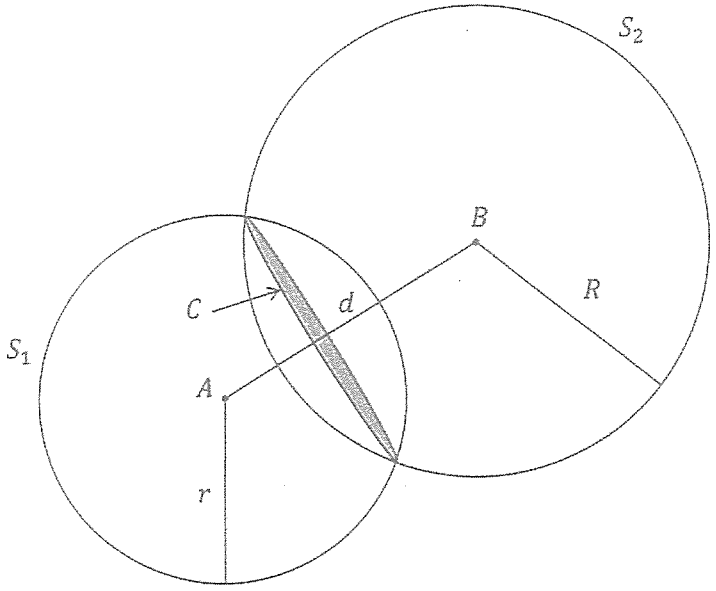
iii. Hence, find the sum of the volume of the two square pyramids $OABCD$ and $OEFGH$. [1]

iv. Show that $\overrightarrow{OA} \cdot \overrightarrow{OC} = -\dfrac{1}{3}$ [1]

(c) Two spheres S_1 and S_2 intersect in the circle C.

The sphere S_1 is centered at A(a,b,c) with radius r and the sphere S_2 is centered at $B(a+m, b+n, c+p)$ with radius R, where a, b, c, n, p, r and R are real, with $r, R > 0$.

The distance between the centres of the spheres is d.



i. Prove that the circle C lies on the sphere S_3, which has AB as a diameter, only if $m^2 + n^2 + p^2 = r^2 + R^2$. [4]

ii. Hence, or otherwise, show that the intersection of the spheres $x^2 + y^2 + z^2 = 25$ and $(x-3)^2 + (y-4)^2 + (z-12)^2 = 144$ is a circle lying on a third sphere, which has the interval joining the centres of the first two spheres as a diameter. [1]

End of Task 2

2 PC for feedback

Suggested Solutions

Question (1) option (C)

$\exists x (F(x) \cap P(x))$

C	C	A	B	D

(2) option (C)

Contrapositive: $f''(c) \neq 0 \Rightarrow f(x)$ does not have a stationary point of inflexion at $x = c$: True

Converse: $f''(c) = 0 \Rightarrow f(x)$ has a stationary point of inflexion at $x = c$.

(3) option (A)

$$|\underset{\sim}{v}| = \sqrt{5^2 + (-3)^2 + 3^2}$$

$$= \sqrt{43}$$

$$\theta_i = \cos^{-1}\left(\frac{5}{\sqrt{43}}\right), \quad \theta_j = \cos^{-1}\left(\frac{-3}{\sqrt{43}}\right); \quad \theta_k = \cos^{-1}\left(\frac{3}{\sqrt{43}}\right)$$

$\therefore \theta_i = 40^\circ 19', \quad \theta_j = 117^\circ 14', \quad \theta_k = 62^\circ 47'$

Marker's Feedback

Suggested Solutions

Question 4 option (B)

Spiralling in yz plane along x-axis. Point (0, 0, 2)

Question 5 option (D)

Since collinear, $\overrightarrow{RS} = \lambda \overrightarrow{ST}$

$$\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda \left(\begin{pmatrix} 3 \\ m \\ n \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \right)$$

$$\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ m+1 \\ n-1 \end{pmatrix} \quad \text{so } \lambda = 1$$

$-2 = m + 1$ $\qquad$ $1 = n - 1$

$m = -3$ $\qquad$ $n = 2$

Marker's Feedback

Suggested Solutions

Negation of $p \to q$ is $p \to \sim q$

Question 6a

Negation: $\exists x \in R$ such that $(x > 1)$ and $(x \geq x^2)$

$\exists x \in R, (x > 1) \cap (x \geq x^2)$ poorly done many students writing $\in$ instead of $\exists$ (not penalised here)

6b Assume $a^2 - 4b - 3 = 0 \quad (a, b \in z)$

So $a^2 - 4ab = 3$

If a is even, $a = 2n \quad (n \in z)$ Some students did not consider odd and even

$(2n)^2 - 4b = 3$

$4n^2 - 4b = 3$

$2(2n^2 - 2b) = 3$

which is a contradiction since LHS is even.

If a is odd, $a = 2n + 1 \quad (n \in z)$

$(2n+1)^2 - 4b = 3$

$4n^2 + 4n - 4b = 2$

$4(n^2 + n - b) = 2$

$n^2 + n - b = \frac{1}{2}$

which is a contraction since LHS must be an integer

$\therefore a, b \in z \Rightarrow$

Marker's Feedback

Suggested Solutions

Question 6C

Suppose $\log_2 n$ is rational

Let $n = 2k+1$ for $k \in \mathbb{Z}^+, \geq 3$

$\therefore \log_2(2k+1) = \frac{p}{q}$ for $p, q \in \mathbb{Z}$, $HCF(p,q) = 1$, $q \neq 0$ → (1)

$\therefore q \log_2(2k+1) = p$

$\log_2(2k+1)^q = p$

$(2k+1)^q = 2^p$ — (2) (1)

LHS is odd, since an odd number to any power is odd

RHS is even, since an even number to any power is even

We have a contraction, so $\log_2 n$ must be irrational for all odd n. — (3) (1)

Marker's Feedback

well done overall.

no reasoning or poor reasoning by many students

The last step must be explained explicitly to achieve full marks.

Suggested Solutions

Question (6d)

$$12^n = (5+7)^n$$

$$= {}^nC_0 5^n + {}^nC_1 5^{n-1} \times 7 + {}^nC_2 5^{n-2} \times 7^2 + \cdots + {}^nC_{n-1} \cdot 5 \times 7^{n-1} + {}^nC_n 7^n$$

As ${}^nC_0 = {}^nC_n = 1$ then

$$(5+7)^n = 5^n + {}^nC_1 5^{n-1} \times 7 + {}^nC_2 5^{n-2} \times 7^2 + \cdots + {}^nC_{n-1} \times 5 \times 7^{n-1} + 7^n$$

Since ${}^nC_1 5^{n-1} \times 7 + {}^nC_2 5^{n-2} \times 7^2 + \cdots + {}^nC_{n-1} 5 \times 7^{n-1}$

is a sum of positive terms

then $(5+7)^n = 5^n + 7^n +$ Sum of positive terms.

This indicates that $(5+7)^n > 5^n + 7^n$; that is $12^n > 5^n + 7^n \ \forall\ n \geq 2$.

Marker's Feedback

Poorly done overall.
Students who knew where to start were generally successful.

(7)(a) Assume $\sqrt[3]{n}$ is rational, n prime

$\therefore \sqrt[3]{n} = \frac{p}{q}$, $p, q \in \mathbb{Z}$, $q \neq 0$.

$n = \frac{p^3}{q^3}$

$nq^3 = p^3$ (1)

$\therefore$ As p, q have no common factors

n is a factor of $p^3 \Rightarrow n$ is a factor of p.

$p = kn$

$p^3 = k^3 n^3$ (2)

$\therefore$ Equating (1) & (2)

$nq^3 = n^3 k^3$

$q^3 = n^2 k^3$

n is a factor of $q^3 \Rightarrow n$ is a factor of q

But p, q have no common factors.

There is a contradiction.

$\sqrt[3]{n}$ is irrational.

Marker's Feedback

Students were often intimidated by the fact that n is a general prime number and the proof is for $\sqrt[3]{n}$ to be irrational rather than $\sqrt{n}$. This is despite the fact that the method & level of difficulty is the same as, for example, proving $\sqrt{5}$ is irrational.

Suggested Solutions

Question 7b

(i) $a^2+b^2 > 2ab$ — (1)

$b^2+c^2 > 2bc$ — (2)

$a^2+c^2 > 2ac$ — (3)

(1)+(2)+(3) $2a^2+2b^2+2c^2 > 2ab+2bc+2ac$

$2(a^2+b^2+c^2) > 2(ab+bc+ac)$

$\therefore a^2+b^2+c^2 > ab+bc+ac$ (as required)

(ii) Since $a+b+c=6$

$(a+b+c)^2=(6)^2$

$((a+b)+c)^2=36$

$a^2+b^2+c^2+2(ab+bc+ac)=36$ — (1)

↳ (4)

from (1) $a^2+b^2+c^2 > ab+bc+ac$

$a^2+b^2+c^2+2(ab+bc+ac) > ab+bc+ac+2(ab+bc+ac)$

Marker's Feedback

(i) Generally well done.

(ii) Also generally reasonably well done.

Suggested Solutions

Question 7b ii, continued

$a^2+b^2+c^2+2(ab+bc+ac) > 3(ab+bc+ac)$ ↳ (5)

sub (4) into (5)

$36 > 3(ab+bc+ac)$

$\therefore ab+bc+ac < 12$ — (1)

Marker's Feedback

— See previous page —

Suggested Solutions

Question 7C

Let the three consecutive even numbers be

$2n, 2n+2, 2n+4$

$S = (2n)^2 + (2n+2)^2 + (2n+4)^2$ — (1)

$= 4n^2 + 4n^2 + 8n + 4 + 4n^2 + 16n + 16$

$= 12n^2 + 24n + 20$

$= 12(n^2 + 2n + 1) + 8$

$= 12(n+1)^2 + 8$

$= 12M + 8$; $M \in \mathbb{N}$ — (1)

∴ The sum of any three consecutive even integers can be divided by 12 and leave a remainder of 8.

Marker's Feedback

Generally well done.
[note: could also use $(2n-2)^2 + (2n)^2 + (2n+2)^2$ which simplifies the algebra].

Suggested Solutions

Question 8a

let p: $n+6$ is a positive odd integer

q: $n+21$ is a positive even integer.

RTP: $p \iff q$ is true i.e we need to prove that $p \Rightarrow q$ and $q \Rightarrow p$ are both true.

Now, let us prove $p \Rightarrow q$; that is

$n+21$ is a positive ~~integer~~ even integer if $n+6$ is a positive odd integer.

Let $n+6$ be a positive odd integer. This means

$n+6 = 2r+1$, where r is a positive integer.

$\therefore n = 2r-5$ with $r > 2.5$

$n+21 = 2r-5+21$

$= 2r+16$

$= 2(r+8)$, which is even

So $n+21$ is a positive even integer if $n+6$ is odd.

Marker's Feedback

Generally OK, but be VERY wary.
With questions like this, NEVER use "ODD + ODD = EVEN" etc.
We are actually asking you to PROVE something like this.
Use ALGEBRA.

→ Some people lost marks for not proving BOTH cases
Remember: if & only if means PROVE
If $n+6$ is odd, $n+21$ is EVEN AND
if $n+21$ is even, $n+6$ is ODD.

Suggested Solutions

Question 8a) Continued

Now let us prove $q \Rightarrow p$, i.e

if $n+21$ is a positive even integer, then $n+6$ is a positive odd integer.

Let $n+21$ be a positive even integer. Then

$n+21 = 2p$, where p is an integer

$n = 2p - 21$; $p > 10.5$

So, $n+6 = 2p - 21 + 6$

$= 2p - 15$

$= 2p - 2 \times 8 + 1$

$= 2(p-8) + 1$, which will be odd for all p and certainly for $p > 10.5$

$\therefore$ $n+21$ is an even positive integer, then $n+6$ is a positive odd integer.

Hence, the logical equivalence is true.

Marker's Feedback

See previous page

Suggested Solutions

Question 8b

$P = 2(a+b)$; $A = ab$

$\frac{a+b}{2} \geq \sqrt{ab}$ — (1)

$\frac{2(a+b)}{4} \geq \sqrt{ab}$ — (1)

$\frac{P}{4} \geq \sqrt{A}$ (3m)

$\frac{P^2}{16} \geq A$

$P^2 \geq 16A$ — (1)

Marker's Feedback

Generally well done.

However, some people [a small minority] proved the converse

if $\frac{P^2}{16} \geq ab$, $\frac{a+b}{2} \geq \sqrt{ab}$.

THIS IS COMPLETELY INVALID AS THE CONVERSE MAY NOT BE TRUE AND GOT 0 MARKS!

[e.g. Proving ~~a square~~ if a quadrilateral is a square it has four 90° angles does NOT mean if it has four 90° angles it's a square].

Suggested Solutions

Question 8c.

Let the prime be p and the square number be n^2.

For $n^2 = 1$ and $p = 2$, $p > n^2$

For $n^2 = 4$ and $p = 3$, $p < n^2$ (exception)

For $n \geq 3$, if $p = n^2 - 1$ then

$p = (n-1)(n+1)$ where, since

$n \geq 3$, $n - 1 > 1$ and $n + 1 > 1$ which implies that p is composite.

This is a contradiction so the situation where the prime is one less than the square is impossible.

Hence, for $n \geq 3$, $p = n^2 + 1$

∴ The conjecture is true.

Marker's Feedback

Very poorly done. Only a few students realised the key step: $n^2 - 1$ can be factorised to $(n-1)(n+1)$, which means that if $n \geq 3$, $n^2 - 1$ is composite.

~~NOTE: PROVING THE CONVERSE!~~

Suggested Solutions

Question

(9a) $\underset{\sim}{u}+\underset{\sim}{v} = 17\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$ — (1)

$\underset{\sim}{u}-\underset{\sim}{v} = \underset{\sim}{i} + 9\underset{\sim}{j} - 4\underset{\sim}{k}$ — (2)

(1) + (2) $2\underset{\sim}{u} = 18\underset{\sim}{i} + 8\underset{\sim}{j} - 2\underset{\sim}{k}$ } (1)

(1) − (2) $2\underset{\sim}{v} = 16\underset{\sim}{i} - 10\underset{\sim}{j} + 6\underset{\sim}{k}$

$\therefore \underset{\sim}{u} = \begin{pmatrix} 9 \\ 4 \\ -1 \end{pmatrix}$ and $\underset{\sim}{v} = \begin{pmatrix} 8 \\ -5 \\ 3 \end{pmatrix}$

$\cos\theta = \dfrac{72 - 20 - 3}{\sqrt{9^2+4^2+(-1)^2}\cdot\sqrt{8^2+(-5)^2+3^2}}$ → (1)

$= \dfrac{1}{2}$

Hence the acute angle between the vectors is $\dfrac{\pi}{3}$. — (1)

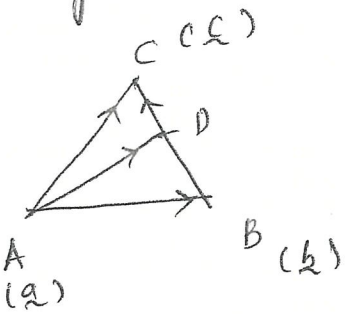
Marker's Feedback

Well done.

Suggested Solutions

Question

(9b) Let 'O' be the Origin and A, B, C have position vectors $\underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{c}$ respectively.



$\overrightarrow{AB} = \underset{\sim}{b} - \underset{\sim}{a}$ and $\overrightarrow{AC} = \underset{\sim}{c} - \underset{\sim}{a}$

Hence $\overrightarrow{BD} = \frac{1}{2}(\underset{\sim}{c} - \underset{\sim}{b})$ — (1)

$\Rightarrow \overrightarrow{OD} = \overrightarrow{OB} + \overrightarrow{BD}$

$= \underset{\sim}{b} + \frac{1}{2}(\underset{\sim}{c} - \underset{\sim}{b})$

$= \frac{1}{2}(\underset{\sim}{c} + \underset{\sim}{b})$

Hence $\overrightarrow{AD} = \frac{1}{2}(\underset{\sim}{b} + \underset{\sim}{c}) - \underset{\sim}{a}$ — (1)

$2\left(|\overrightarrow{AD}|^2 + |\overrightarrow{BD}|^2\right) = 2\left[\left(\frac{1}{2}(\underset{\sim}{b}+\underset{\sim}{c}) - \underset{\sim}{a}\right)\cdot\left(\frac{1}{2}(\underset{\sim}{b}+\underset{\sim}{c}) - \underset{\sim}{a}\right) + \left(\frac{1}{2}(\underset{\sim}{c}-\underset{\sim}{b})\right)\cdot\left(\frac{1}{2}(\underset{\sim}{c}-\underset{\sim}{b})\right)\right]$

$= 2\left[\frac{1}{4}(\underset{\sim}{b}+\underset{\sim}{c})\cdot(\underset{\sim}{b}+\underset{\sim}{c}) - \underset{\sim}{a}\cdot(\underset{\sim}{b}+\underset{\sim}{c}) + \underset{\sim}{a}\cdot\underset{\sim}{a} + \frac{1}{4}\underset{\sim}{c}\cdot\underset{\sim}{c} - \frac{1}{2}\underset{\sim}{b}\cdot\underset{\sim}{c} + \frac{1}{4}\underset{\sim}{b}\cdot\underset{\sim}{b}\right]$

Marker's Feedback

You could also use $\overrightarrow{DC} = -\overrightarrow{DB}$

This can lead to $\overrightarrow{AB} = \overrightarrow{AD} + \overrightarrow{DB}$ & $\overrightarrow{AC} = \overrightarrow{AD} - \overrightarrow{DB}$

So as $|\overrightarrow{AB}|^2 = \overrightarrow{AB}\cdot\overrightarrow{AB} = (\overrightarrow{AD}+\overrightarrow{DB})\cdot(\overrightarrow{AD}+\overrightarrow{DB}) = |\overrightarrow{AD}|^2 + |\overrightarrow{DB}|^2 + 2\overrightarrow{AD}\cdot\overrightarrow{DB}$

& similarly $|\overrightarrow{AC}|^2 = |\overrightarrow{AD}|^2 + |\overrightarrow{DB}|^2 - 2\overrightarrow{AD}\cdot\overrightarrow{DB}$

$|\overrightarrow{AB}|^2 + |\overrightarrow{AC}|^2 = 2(|\overrightarrow{AD}|^2 + |\overrightarrow{DB}|^2)$

Not very well done.

Suggested Solutions

Question (9b) continued.

$$= 2\left[\frac{1}{4}\underset{\sim}{b}\cdot\underset{\sim}{b} + \frac{1}{2}\underset{\sim}{b}\cdot\underset{\sim}{c} + \frac{1}{4}\underset{\sim}{c}\cdot\underset{\sim}{c} - \underset{\sim}{a}\cdot\underset{\sim}{b} - \underset{\sim}{a}\cdot\underset{\sim}{c} + \underset{\sim}{a}\cdot\underset{\sim}{a} + \frac{1}{4}\underset{\sim}{c}\cdot\underset{\sim}{c} - \frac{1}{2}\underset{\sim}{b}\cdot\underset{\sim}{c} + \frac{1}{4}\underset{\sim}{b}\cdot\underset{\sim}{b}\right]$$

$$= 2\left[\frac{1}{2}\underset{\sim}{b}\cdot\underset{\sim}{b} + \frac{1}{2}\underset{\sim}{c}\cdot\underset{\sim}{c} - \underset{\sim}{a}\cdot\underset{\sim}{b} - \underset{\sim}{a}\cdot\underset{\sim}{c} + \underset{\sim}{a}\cdot\underset{\sim}{a}\right] \quad (1)$$

$$= \underset{\sim}{b}\cdot\underset{\sim}{b} + \underset{\sim}{c}\cdot\underset{\sim}{c} - 2\underset{\sim}{a}\cdot\underset{\sim}{b} - 2\underset{\sim}{a}\cdot\underset{\sim}{c} + 2\underset{\sim}{a}\cdot\underset{\sim}{a}$$

$$= \underset{\sim}{b}\cdot\underset{\sim}{b} - 2\underset{\sim}{a}\cdot\underset{\sim}{b} + \underset{\sim}{a}\cdot\underset{\sim}{a} + \underset{\sim}{c}\cdot\underset{\sim}{c} - 2\underset{\sim}{a}\cdot\underset{\sim}{c} + \underset{\sim}{a}\cdot\underset{\sim}{a}$$

$$= (\underset{\sim}{b} - \underset{\sim}{a})\cdot(\underset{\sim}{b} - \underset{\sim}{a}) + (\underset{\sim}{c} - \underset{\sim}{a})\cdot(\underset{\sim}{c} - \underset{\sim}{a})$$

$$= |\overrightarrow{AB}|^2 + |\overrightarrow{AC}|^2 \quad - (1)$$

Marker's Feedback

See previous.

Suggested Solutions

Question

(9c) i) Since L_1 and L_2 are $\perp$ then

$$\begin{pmatrix} -2 \\ 1 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} q \\ 2 \\ 2 \end{pmatrix} = 0$$

$$\therefore \quad -2q + 2 - 8 = 0 \qquad \text{— (1m)}$$

$$\Rightarrow q = -3$$

ii) As L_1 and L_2 intersect, then

$$11 - 2\lambda = -5 - 3\mu \quad \text{— (1)}$$

$$2 + \lambda = 11 + 2\mu \quad \text{— (2)} \qquad \} \ (1m)$$

$$17 - 4\lambda = p + 2\mu \quad \text{— (3)}$$

(1) + 2 × (2) $\Rightarrow$ $15 = 17 + \mu \Rightarrow \mu = -2$ } (1m)

From (2) $2 + \lambda = 11 - 4 \Rightarrow \lambda = 5$ and

From (3) $17 - 20 = p - 4 \Rightarrow p = 1$ → (1m)

Marker's Feedback

Most people knew what to do, but there were a LOT of silly mistakes.

Suggested Solutions

Question

9c (iii)

At the point of intersection

$x = 11 - 2\lambda$; $11 - 2(5) = 1$

$y = 2 + \lambda$; $2 + 5 = 7$

$z = 17 - 4\lambda$; $17 - 4(5) \Rightarrow -3$

} some working out (1m)

Hence the lines intersect at $(1, 7, -3)$

↳ point (1m)

Marker's Feedback

Suggested Solutions

Question 10a

(i) $P(3,2,1)$ lies on S_1, $C(2,3,4)$

$\therefore PQ$ is diameter

$\therefore \overrightarrow{PC} = \overrightarrow{CQ}$

$\therefore Q(a,b,c) = (1,4,7) \rightarrow (1)$

(ii) $\underset{\sim}{r} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + t\begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}$

$P(3,2,1)$ lies on S_1. Checking directions

$$\begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix} \cdot \overrightarrow{PC} = \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{bmatrix} 2-3 \\ 3-2 \\ 4-1 \end{bmatrix}$$

$$= \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$$

$$= 4(-1) + (1) + (3)$$

$$= 0$$

$\therefore \underset{\sim}{r}$ is tangent.

Marker's Feedback

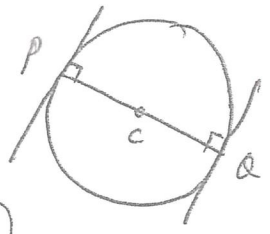
Many students forget that a tangent is perpendicular to a radius.

Suggested Solutions

Question

10aiii

$\underset{\sim}{L} \perp \underset{\sim}{r}$; $\overrightarrow{PQ} \perp \underset{\sim}{L}$



let $\underset{\sim}{L} = \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix} + \mu \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

So $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix} = 0$ and $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} = 0$ — (1)

↳ (1eq) ↳ (2eq)

$\therefore 4a+b+c=0$; $-a+b+3c=0$

(1) − (2) $5a-2c=0$ $\therefore 2c = 5a$

let $c=5$ and $a=2$ $\therefore b=-13$

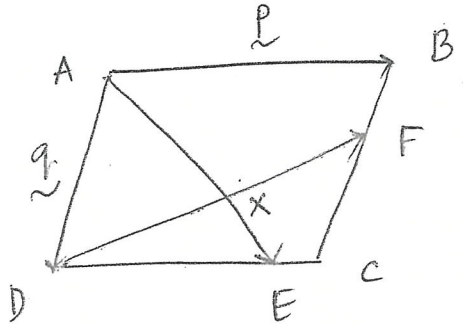
$\therefore \underset{\sim}{L} = \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -13 \\ 5 \end{pmatrix}$ — (1)

Marker's Feedback

Finding a line perpendicular to two other lines was very poorly done.

Suggested Solutions

Question 10b



(i) Proof:

$\overrightarrow{AX} = \mu \overrightarrow{AE}$

$\overrightarrow{AX} = \mu \overrightarrow{AE}$

$= \mu \left(\overrightarrow{AD} + \frac{3}{4} \overrightarrow{DC} \right)$ — (1)

$= \mu \left(\underset{\sim}{q} + \frac{3}{4} \underset{\sim}{p} \right)$ — (1)

Marker's Feedback

(i) Reasonably well done.

Suggested Solutions

Question 10b ii)

From i), $\overrightarrow{AX} = \mu\left(\underset{\sim}{q} + \frac{3}{4}\underset{\sim}{p}\right)$

$\overrightarrow{AX} = \underset{\sim}{q} + \overrightarrow{DX}$

$\overrightarrow{DX} = \overrightarrow{AX} - \underset{\sim}{q}$

$\overrightarrow{DX} = \mu\left(\underset{\sim}{q} + \frac{3}{4}\underset{\sim}{p}\right) - \underset{\sim}{q}$

$\overrightarrow{DX} = (\mu - 1)\underset{\sim}{q} + \frac{3}{4}\mu\underset{\sim}{p}$ — (1) 1m

$\overrightarrow{DX} = \lambda\overrightarrow{DF}$

$= \lambda\left(\overrightarrow{DC} + \overrightarrow{CF}\right)$

$= \lambda\left(\underset{\sim}{p} + \frac{2}{3}\overrightarrow{CB}\right)$

$= \lambda\left(\underset{\sim}{p} - \frac{2}{3}\underset{\sim}{q}\right)$

$= \lambda\underset{\sim}{p} - \frac{2}{3}\lambda\underset{\sim}{q}$ — (2) 2m

Marker's Feedback

Not well done & a LOT of fudging.
A lot of silly mistakes as well.
Note: We only need to prove $\lambda = \frac{1}{2}$ [not μ] as AE bisects DX.
DX does NOT bisect AE.

Suggested Solutions

Question 10b ii) Continued

equating (1) & (2)

$$(\mu-1)\underset{\sim}{q}+\frac{3}{4}\mu\underset{\sim}{p}=\lambda\underset{\sim}{p}-\frac{2}{3}\lambda\underset{\sim}{q}$$

$\underset{\sim}{q}$ & $\underset{\sim}{p}$ are not parallel

$\therefore$ Equating ~~coefficients of~~ q_2's & P_S

$$(\mu-1)\underset{\sim}{q}=-\frac{2}{3}\lambda\underset{\sim}{q}$$

$$(1-\mu)\underset{\sim}{q}=\frac{2}{3}\lambda\underset{\sim}{q}$$

$$1-\mu=\frac{2}{3}\lambda$$

sub $\lambda=\frac{3}{4}\mu$ here

$$1-\mu=\frac{2}{3}\times\frac{3}{4_2}\mu$$

$$1-\mu=\frac{1}{2}\mu \quad \therefore \mu=\frac{2}{3}$$ (1M)

$$\frac{3}{4}\mu\underset{\sim}{p}=\lambda\underset{\sim}{p}$$

$$\lambda=\frac{3}{4}\mu$$

and $\lambda=\frac{3}{4}\times\frac{2}{3}=\frac{1}{2}$

Marker's Feedback

See previous page

Suggested Solutions

Question 10b, ii, continued

$\therefore \overrightarrow{DX} = \frac{1}{2}\overrightarrow{DF}$ X is mid point

$\therefore \overrightarrow{AF}$ bisects DF.

Marker's Feedback

See 2 previous pages

Suggested Solutions

Easiest way to handle (10)(c) is to PLOT POINTS.

t	0	$t=\frac{\pi}{2}$	$t=\pi$
x	$e^0\cos 0 = e^0 = 1$	$x = e^{\frac{\pi}{2}}\cos\frac{\pi}{2} = 0$	$x = e^{\pi}\cos\pi = -e^{\pi}$
y	$e^0\sin 0 = 0$	$y = e^{\frac{\pi}{2}}\sin\frac{\pi}{2} = e^{\frac{\pi}{2}}$	$y = e^{\pi}\sin\pi = 0$
Point	$(1,0)$	$\therefore$ Point $= (0, e^{\frac{\pi}{2}})$	Point $= (-e^{\pi}, 0)$

So the curve is a spiral, spiraling ANTI-CLOCKWISE OUT FROM THE ORIGIN.

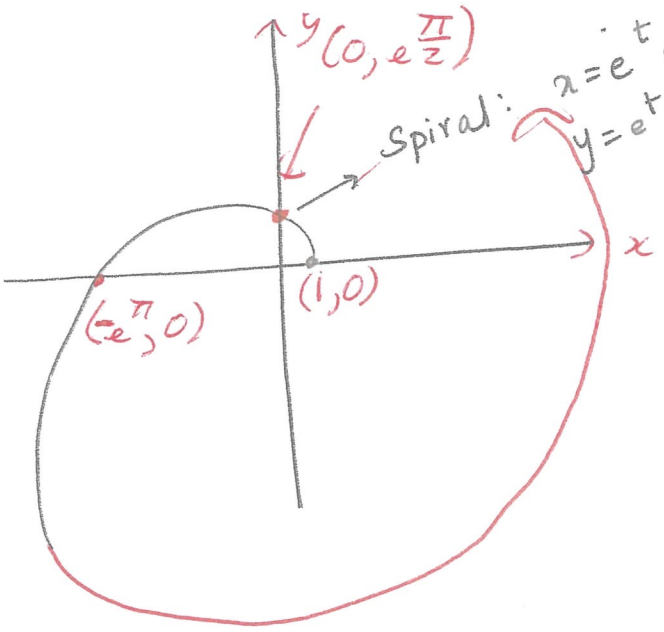
Marker's Feedback

REALLY poorly done.
The best way to sketch hard 20 or 30 curves is to plot points.

Suggested Solutions

Question

10(ii),



(1)

Marker's Feedback

See previous page

Suggested Solutions

Question

(11a) From the diagram

$\overrightarrow{PA} = \overrightarrow{PS} + \overrightarrow{SA}$ and $\overrightarrow{PC} = \overrightarrow{PS} + \overrightarrow{SC}$

$\therefore \overrightarrow{PA} + \overrightarrow{PC} = \overrightarrow{PS} + \overrightarrow{SA} + \overrightarrow{PS} + \overrightarrow{SC}$

Now as $\overrightarrow{SA}$ and $\overrightarrow{SC}$ are equal radii, but in opposite directions

$\therefore \overrightarrow{SC} = -\overrightarrow{SA}$

$\therefore \overrightarrow{PA} + \overrightarrow{PC} = \overrightarrow{PS} + \overrightarrow{SA} + \overrightarrow{PS} - \overrightarrow{SA}$

$= 2\overrightarrow{PS}$ ——— (eq1)

Similarly $\overrightarrow{PB} + \overrightarrow{PD} = 2\overrightarrow{PS}$ → (eq2)

Also, $\overrightarrow{PE} = \overrightarrow{PR} + \overrightarrow{RE}$ and $\overrightarrow{PG} = \overrightarrow{PR} + \overrightarrow{RG}$

Now, as $\overrightarrow{RE}$ and $\overrightarrow{RG}$ are equal radii, but in opposite directions, this means

$\overrightarrow{RG} = -\overrightarrow{RE}$

Marker's Feedback

well done overall.
Various methods used.
Some Students tried to use magnitude but this was not accepted.

Suggested Solutions

Question 11a continued

$\overrightarrow{PE} + \overrightarrow{PG} = \overrightarrow{PR} + \overrightarrow{RE} + \overrightarrow{PR} - \overrightarrow{RE}$

$= 2\overrightarrow{PR}$ → eq 3

Similarly $\overrightarrow{PF} + \overrightarrow{PH} = 2\overrightarrow{PR}$ → eq 4

Now eq (1) + eq (2) + eq (3) + eq (4)

$\overrightarrow{PA} + \overrightarrow{PB} + \overrightarrow{PC} + \overrightarrow{PD} + \overrightarrow{PE} + \overrightarrow{PF} + \overrightarrow{PG} + \overrightarrow{PH}$

$= \overrightarrow{PA} + \overrightarrow{PC} + \overrightarrow{PB} + \overrightarrow{PD} + \overrightarrow{PE} + \overrightarrow{PG} + \overrightarrow{PF} + \overrightarrow{PH}$

$= 2\overrightarrow{PS} + 2\overrightarrow{PS} + 2\overrightarrow{PR} + 2\overrightarrow{PR}$

$= 4\overrightarrow{PS} + 4\overrightarrow{PR}$ eq 5

Now, as RS is a straight line and PR = h and PS = 2h,

$\therefore 2\overrightarrow{PR} = -\overrightarrow{PS}$

Sub in eq (5), we get

$\overrightarrow{PA} + \overrightarrow{PB} + \overrightarrow{PC} + \overrightarrow{PD} + \overrightarrow{PE} + \overrightarrow{PF} + \overrightarrow{PG} + \overrightarrow{PH} = 4\overrightarrow{PS} - 2\overrightarrow{PS}$

$= 2\overrightarrow{PS}$

Marker's Feedback

Suggested Solutions

Question

(ii b i) Let $(0,0,0)$ and (a,a,a) be the points which form the longest diagonal in a cube with side length 'a' units.

$$\text{length} = \sqrt{a^2+a^2+a^2} = \sqrt{3a^2} = \sqrt{3}\,a \quad \square \rightarrow (1m)$$

well done

(ii b ii) The longest diagonal is AG which is also the diameter of the sphere.

Hence, $AG = 2$

$2 = \sqrt{3}\,a$ (1)m

$a = \frac{2}{\sqrt{3}}$ D

well done

Marker's Feedback

Suggested Solutions

Question

11biii

6 square pyramids are required to form the entire volume of the cube.

$\therefore$ Volume $= \frac{2}{6} \times a^3$

$= \frac{1}{3}\left(\frac{2}{\sqrt{3}}\right)^3$

$= \frac{8\sqrt{3}}{27}$ — (1)

Many did not realise they needed volume of 2 pyramids

11biv point A $\left(-\frac{1}{2}a, \frac{1}{2}a, \frac{1}{2}a\right)$ and C $\left(\frac{1}{2}a, \frac{-1}{2}a, \frac{1}{2}a\right)$

$\overrightarrow{DA} \cdot \overrightarrow{DC} = \frac{-1}{4}a^2 - \frac{1}{4}a^2 + \frac{1}{4}a^2$

$= \frac{-1}{4}a^2$

$= \frac{-1}{4}\left(\frac{4}{3}\right)$

$= \frac{-1}{3}$ → (1m)

Poorly done.

Marker's Feedback

Suggested Solutions

Question

(11 c i)

$$d^2 = |\overrightarrow{AB}|^2 = (a+m-a)^2 + (b+n-b)^2 + (c-p-c)^2$$
$$= m^2+n^2+p^2 \quad \text{— (eq 1)}$$

Method 1: If $m^2+n^2+p^2 = r^2+R^2$

$$d^2 = r^2+R^2 \rightarrow \text{(eq 2)}$$

Let P be any point on C.

From (2): $|\overrightarrow{AB}|^2 = |\overrightarrow{AP}|^2 + |\overrightarrow{PB}|^2$ so ΔAPB is right angled at P

$\therefore \overrightarrow{AB} \cdot \overrightarrow{PB} = 0$

$\therefore$ P lies on the sphere S_3 with diameter AB.

Since P is any point on C then C lies on S_3 if $m^2+n^2+p^2 = r^2+R^2$

Marker's Feedback

many non attempts

Suggested Solutions

Question 11 c i continued

Method 2: if $m^2+n^2+p^2 \neq r^2+R^2$

$d^2 \neq r^2+R^2 \longrightarrow$ (eqn 3)

$\therefore |\overrightarrow{AB}|^2 \neq |\overrightarrow{AP}|^2 + |\overrightarrow{PB}|^2$ so ΔAPB is not right angled at P.

$\therefore \overrightarrow{AP} \cdot \overrightarrow{PB} \neq 0$

$\therefore$ P does not lie on the sphere S_3.

Since P is any point on 'C' then 'C' does not lie on S_3 if $m^2+n^2+p^2 \neq r^2+R^2$

$\therefore$ the circle 'C' lies on the sphere S_3 whose diameter is AB

only if $m^2+n^2+p^2 = r^2+R^2$

Marker's Feedback

Suggested Solutions

Question 11 (i)

$x^2+y^2+z^2=25$ is a sphere with centre $(0,0,0)$ and radius 5 $(r=5)$

$(x-3)^2+(y-4)^2+(z-12)^2=144$ is a sphere with centre $(3,4,12)$ and radius 12

So $m=3$, $n=4$, $p=12$ & $R=12$

$$m^2+n^2+p^2=3^2+4^2+12^2=169$$

$$r^2+R^2=5^2+12^2=169$$

$$\therefore m^2+n^2+p^2=r^2+R^2$$

So from (i), we can see that the two spheres intersect in a circle that lies on a sphere with a diameter whose ends are at the centres of the spheres.

Marker's Feedback

Those who attempted this question were mostly successful.